Make Computer Arithmetic Great Again?

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An apparent contradiction

- low number of paper submissions to Arith these last ≈ 5 years
- few PhD defenses, few academic positions (at least in North America)

  and yet...

  - just in the last 9 months, good arithmetic-related papers in IEEE Trans. VLSI, IEEE Trans. Computers, ACM TOMS, Mathematics in Computer Science, Numerical Algorithms, Mathematics of Computation, BIT...by authors who don't or rarely frequent the Arith conferences;
  - people in the industry still design arithmetic operators, they also have new needs: deep learning, certified and /or reproducible calculations (e.g. for automated transportation), mixed precision...

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Necessary expertise in related areas:

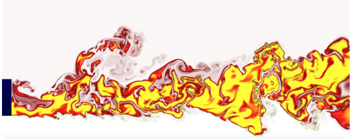
- our community still has expertise on circuit design;
- we have low expertise in numerical analysis, compilation, formal proof, finite field arithmetic...

The domain changed: we used to have

- a few formats: single precision, double precision;
- a few applications: numerical simulation, financial calculations.

Is there a thing such as an "Universal solution"?

Numerical simulation



Embedded computing

Entertainment

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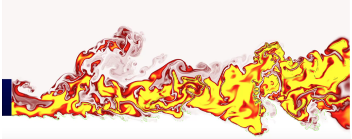

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- floating-point (dynamic range, speed, accuracy)
- crash? just start again the simulation (but not too often)

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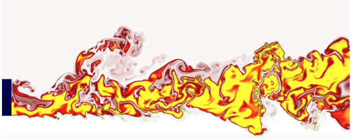

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$\rightarrow$ certified calculations.

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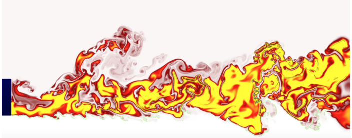

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Entertainment

- supermario’s pizza does not need to carefully follow the laws of physics;
- fluidity matters.

Playing with different formats ?

See Nick Higham's talk at Arith 24.

$$\begin{array}{l}\text{single precision (a.k.a. binary32)}\\ \text{double precision (a.k.a. binary64)}\end{array} \Rightarrow \left\{\begin{array}{l}\text{exotic ad-hoc formats}\\ \text{binary16}\\ \text{binary32}\\ \text{binary64}\\ \text{binary128 (quad)}\end{array}\right.$$

Combinatorial explosion of all the possible arithmetic operators of the form `Format 1` $\times$ `Format 2` $\rightarrow$ `Format 3`.

Cleverly using these formats:

Locate when low precision puts us at an unacceptable risk.	$\longleftrightarrow$	Locate when big precision totally destroys performance.
Numerical analysis abstract interpretation		Compilation Computer architecture

Reproducible computing: useful and sometimes dangerous

Reproducibility in computer arithmetic: examples are

- Reproducible Basic Linear Algebra Subprograms at Berkeley (`https://bebop.cs.berkeley.edu/reproblas/`);
- paper on reproducible summation by P. Ahrens, H.D. Nguyen and J. Demmel.

Main arguments in favor of reproducibility:

- consistence (parallelism: sometimes evaluating the same expression in different places is cheaper than transmitting it);
- debugging is difficult if we cannot reproduce errors;
- contractual/legal reasons.

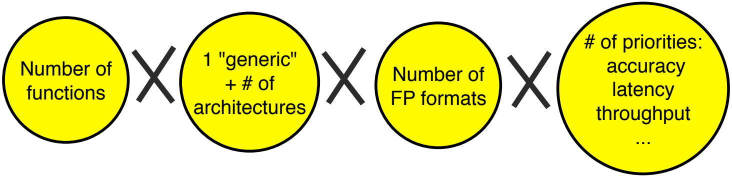
Significant demand (from HPC) and interesting problems $\rightarrow$ need to work on these issues.

Reproducible computing: useful and sometimes dangerous

However. . .

- obtaining very different results when running the same program twice is a sign that something weird is going on (of physical, numerical or programming origin). This is an useful warning, not to be disabled systematically;
- the legal reasons are fine, but there may as well be legal reasons against reproducibility: be ready to explain to a court that you deliberately delivered a less accurate result.

Libraries of math functions



= thousands of function programs

- impossible to debug, maintain, keep consistent, improve...
- and physicists would like many other functions

First solution: computer-assisted library design

Metalibm project (`http://www.metalibm.org`). Two versions

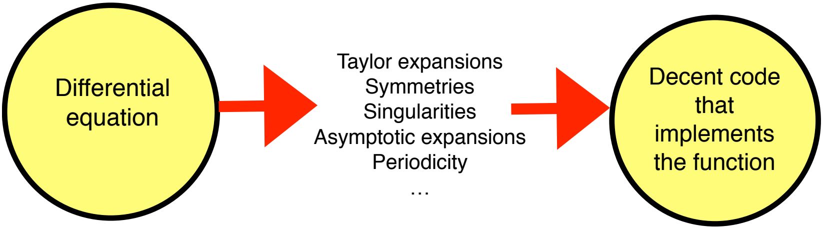
- fully automated for the end user;
- assistance for the specialist.

Metalibm builds upon tools such as

- Sollya (`http://sollya.gforge.inria.fr`): get approximations with many possible constraints;
- Gappa (`http://gappa.gforge.inria.fr`): tight bounds to polynomial evaluation errors and formal proofs.

But this is not the ultimate goal

Tools from computer algebra



- NumGfun: a Package for Numerical and Analytic Computation with D-finite Functions. ISSAC 2010.
- Mezzarobba's PhD dissertation *Autour de l'évaluation numérique des fonctions D-finies* (Ecole Polytechnique, Paris, 2011);
- complex mathematics but big reward.

Incidentally, people from computer algebra need us to speed up various algebraic computations.

Tools from computer algebra

Dynamic Dictionary of Mathematical Functions:

`http://ddmf.msr-inria.inria.fr/1.9.1/ddmf`

The Special Function $J_2(x)$

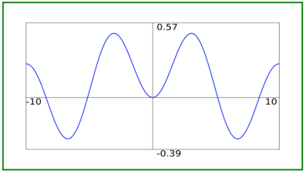
1. Differential Equation

The function $J_2(x)$ satisfies the differential equation

$$x^2 \frac{d^2}{dx^2} y(x) + x \frac{d}{dx} y(x) + \left(x^2 - 4\right) y(x) = 0$$

with initial values $y''(0) = 1/4$ and $y^{(4)}(0) = -1/4$.

2. Plot



min = -10 max = 10 Envoyer

3. Derivative in Terms of Lower-Order Derivatives

$$\frac{d^6}{dx^6} J_2(x) = \frac{\left(-x^6 + 21\,x^4 - 420\,x^2 + 2520\right) J_2(x) - 3\,x\left(x^4 - 35\,x^2 + 420\right) \frac{d}{dx} J_2(x)}{x^6}.$$

order = 6 Envoyer

4. Taylor Expansion at 0

Taylor coefficients:

$$J_2(x) = \frac{1}{4} \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+2}}{4^n n!\,(n+2)!}.$$

Generation of functions at compile-time

- take into account the exact context: underlying architecture, accuracy requirements, priorities (latency/throughput);
- possibly, information on input domain ($\rightarrow$ simplify/avoid range reduction), or special cases (e.g., infinities, NaNs known not to happen);
- compound functions: if you need

$$E_4(x) = \frac{x}{e^x - 1} - \ln(1 - e^{-x}),$$

  then you directly generate $E_4(x)$ instead of generating exp, ln and combining them.
- formal proof absolutely necessary (no library to heavily test beforehand);
- need to work with people from mathematics, computer algebra, compilation, formal proof...

Alternate number representations?

I am not a big fan of Unums, but I reckon J. Gustafson has a point: the considerable increase, in the last 20 years, of the ratio

$$\frac{\text{time to read/write in memory}}{\text{time to perform } +, \times, \div, \sqrt{}}$$

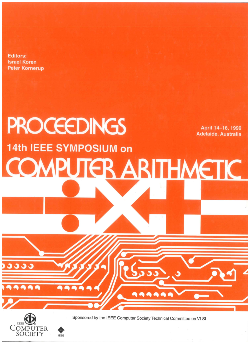
should be viewed as an interesting challenge, not as a sign that we have become useless.

- we have time to do "more" things
- attach easy-to-compute additional information to FP numbers?
- develop communication-avoiding algorithms.

Other topics of interest

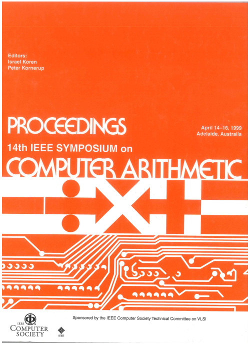
- Approximate computing. Requires more science than computing “exactly”: estimate largest errors, average errors, probabilities of failures, make sure branches are taken consistently, …
- Complex arithmetic that is i. accurate; ii. fast, and iii. overflow/underflow-safe;
- large interchange formats (we will more and more have to deal with data from sensors, previous computations, databases, …);
- hide divisions (this is not only the compiler’s task: for instance one can choose rational approximations that help);
- one day, the quantum computer will be here. Do you want some physicist to re-invent the carry-skip adder or the Dadda multiplier for it?
- …

Time travel



Arith 14, 1999.

Time travel



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Some sessions:

- Addition
- Division
- Divide and Square root
- Multiplication and Rounding
- CORDIC algorithms

Quite a change, isn't it?

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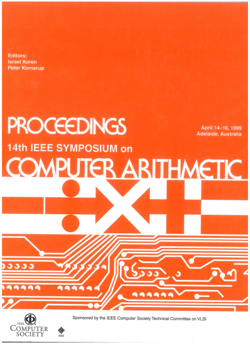

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Need to anticipate the changes: invited talks, PC, ...