A CORRECTLY ROUNDED MIXED-RADIX FUSED-MULTIPLY-ADD

ARITH25 - Amherst, USA
June 25th, 2018

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Motivations

```
int main() {
  _Decimal64 a = 0.1D;
  double b = 10.25;
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- the result is $\mathtt{d} = \mathtt{0x1p} - 52 \approx 2.2204 \cdot 10^{-16}$
- as a reminder, the smallest subnormal number is $\mathtt{0x1p} - 1074 \approx 4.9407 \cdot 10^{-324}$

IEEE 754-2008 - FP formats

Binary format

$(-1)^s \cdot 2^E \cdot m$

s	E	m
1 bit	W_E bits	$p - 1$ bits

Example, `binary64` format:

- significand: $2^{52} \leq m \leq 2^{53} - 1$
- exponent: $-1074 \leq E \leq 971$ (with subnormals)

Decimal format

$(-1)^s \cdot 10^F \cdot n$

s	F	n
1 bit	$w + 5$ bits	$10 \times J$ bits

Example, `decimal64` format:

- significand: $1 \leq n \leq 10^{16} - 1$
- exponent: $-398 \leq F \leq 369$
- binary BID encoding

IEEE 754-2008 - Rounding Modes

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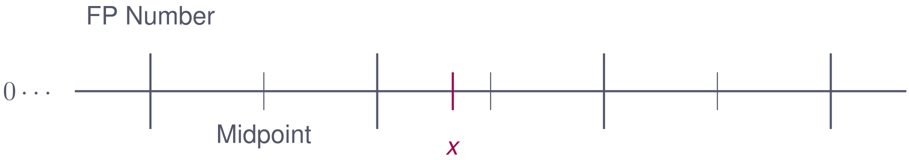

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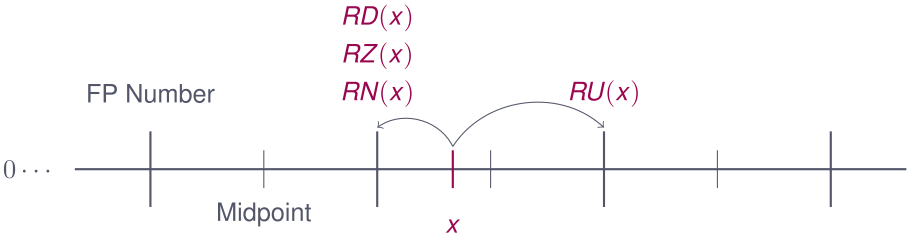

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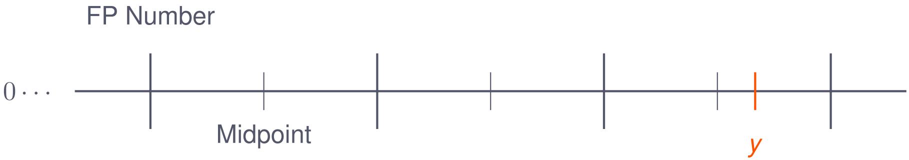

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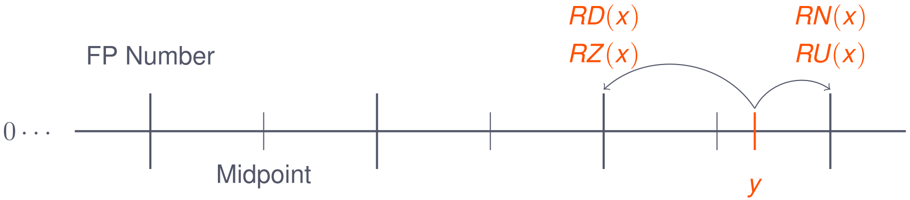

IEEE 754-2008 - Arithmetic Operations

Definitions and properties

- basic arithmetic operations ($+$, $\times$, $\div$, *FMA*...)
- exceptions and flags
- heterogenous operations
  - same base, different format/precision
  - e.g. $\texttt{binary32} = \texttt{binary32} \times \texttt{binary64}$

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Goal: mixed-radix operations

Enrich the IEEE 754-2008 standard with heterogenous operations in base 2 and 10.

Fused Multiply and Add

Definition

$$\mathtt{FMA}(a, b, c) = \circ(a \times b + c) \qquad \mathtt{where}\ \circ \in \{\mathrm{RN}, \mathrm{RZ}, \mathrm{RU}, \mathrm{RD}\}$$

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  - assuming we can represent the midpoint between two FP-numbers
  - compromise between efficiency and implementation effort, e.g. for `binary64` and `decimal64` combinations:
    - 5 operations $+, -, \times, \div, \sqrt{}$ in 20 mixed-radix versions
    - 1 FMA operation in 10 versions

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Goal: mixed-radix FMA

- an emerging need for mixed-radix arithmetic
- implementation of all basic arithmetic operations with one slightly more precise FMA

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Example: consider the exact transcendental number $y = e^x$ and the computed result $\widehat{y} = \exp(x)$.

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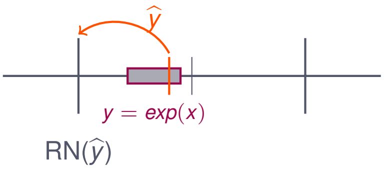

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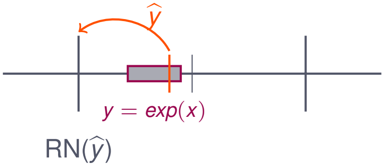

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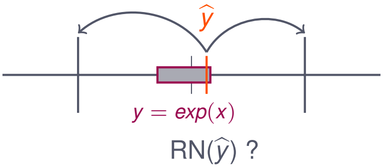
Correct Rounding in the hard case



- not enough accuracy, but how much?

Classical Binary FMA

Algorithm 1 Binary FMA $d = \circ(a \times b + c)$

1: **if** $\frac{a \times b}{c} \notin \left[\frac{1}{2}, 2\right]$ **then**
2: d = *farpath*_addition($a \times b, c$)
3: **else**
4: d = *nearpath*_subtraction($a \times b, c$)
5: **end if**

far-path addition

- when $\frac{a \times b}{c} \notin \left[\frac{1}{2}, 2\right]$
- simple logic with sticky guard bit

near-path subtraction

- when $\frac{a \times b}{c} \in \left[\frac{1}{2}, 2\right]$
- **Sterbenz's lemma**: $(a \times b) - c$ is exactly representable

Mixed-Radix Inexact Cancellation Cases

near-path **subtraction is INEXACT!**

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Obervation

Mixed-radix addition almost always inexact.

Overcoming the TDM

Obervations

- 10 is divisible by 2
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Mixed-Radix unified format

`binary64` and `decimal64` formats can be unified as

$$2^J \cdot 5^K \cdot r$$

with $2^{54} \leq |r| < 2^{55};\ r \in \mathbb{Z}$

$$-1130 \leq J \leq 969;\ -421 \leq K \leq 385;$$

$$J, K \in \mathbb{Z}.$$

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Bound on the worst case of cancellation

- occurs when $(a \times b) - c$ is relatively small
- if $a \times b = 2^L \cdot 5^M \cdot s$ and $c = 2^N \cdot 5^P \cdot t$

$$\left| \frac{s}{t} - 2^{N-L} \cdot 5^{P-M} \right| \geq \eta = 2^{-177.61}$$

- computed using one sided approximations

Performances issues of this exact addition

Size of the accumulator

- acutal computation $\alpha = (a \times b) + c - f$
- a, b and c inputs of the FMA, $(a \times b)$ the exact multiplication bounded into the internal mixed-radix format
- f the closest midpoint bounded into the internal mixed-radix format
- We are sure that we can compute α and store it on 4225 bits, that is 67 words of 64 bits, leaving 63 "free" bits.

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Obervation

In a lot of cases, a quick and not so accurate addition can be enough to perform correct rounding in the output format.

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FMA Mixed Radix Algorithm

Algorithm 2 Mixed-Radix FMA $d = \circ(a \times b + c)$

Multiplication $\psi \leftarrow a \times b$
if it is an "addition" or $\frac{\psi}{c} \notin \left[\frac{1}{2}; 2\right]$ **then**
    $\phi \leftarrow$ *"far-path"* binary addition
else
    $\phi \leftarrow$ *"near-path"* binary subtraction
end if
$\rho \leftarrow$ Conversion of ϕ to the output format
if ρ can round correctly **then**
    return $d \leftarrow \rho$ correctly rounded to output format
else
    Compute integer rounding boundary significand f
    $\alpha \leftarrow$ Exact decimal addition
    Correct ρ using f and the sign of α
    return $d \leftarrow \rho$ correctly rounded to output format
end if

Test Environment and Reference implementations

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GNU Multiple Precision Library (GMP)

- mixed-radix FMA designed in a limited timeframe
- using GMP rational numbers
- Goal: reasonabely fast but easy to design

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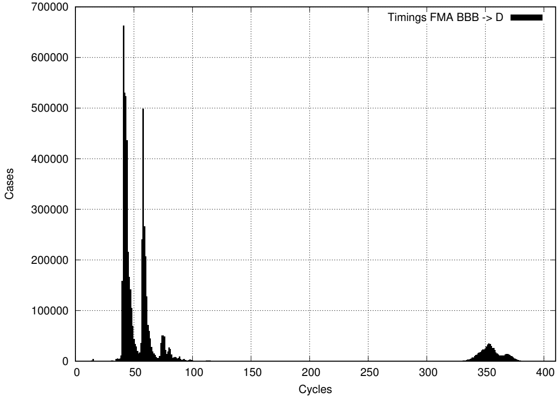
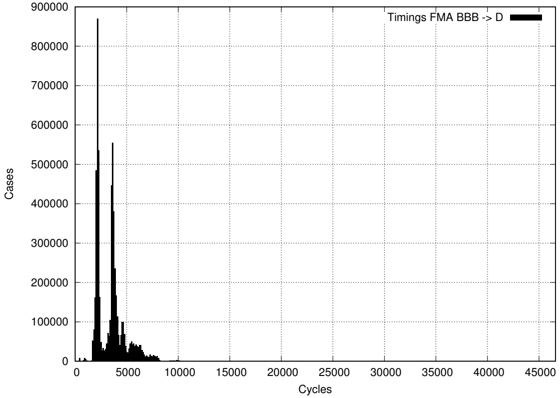
- exact representation of numerical expressions
- evaluated at any precision without spurious rounding

Performance Testing



Our implementation



GMP reference implementation

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Conclusion and Perspectives

Correctly Rounded Mixed-Radix FMA

- two formats: `binary64` and `decimal64`
- pen and paper proof of the algorithm
- overcoming the TDM and worst case of cancellation in the mixed-radix case
- implementation faster than expected and extensively tested

Going further

- more formats!
- mixed-radix FMA of heterogenous precision

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Thank you! Questions?